Book review

Independence-Friendly Logic by Allen L. Mann, Gabriel Sandu and Merlijn Sevenster, 216 pages, Cambridge University Press, 2011. Paperback, ISBN 9780521149341.

Between the two world wars two Polish mathematicians, Stephan Banach and Stanisław Mazur, contemplated on the usefulness of infinite games in mathematics, especially topology (9). In the late fifties the mathematical concept of a game entered mathematical logic. Andrzej Ehrenfeucht introduced his game for the elementary equivalence of two models (1), and in the symposium "Infinitistic Methods", held in Warsaw in 1959, Leon Henkin read a paper (5) suggesting the use of infinite games in infinitary logic.

In both Henkin's game and Ehrenfeucht's game the meaning of the quantifiers $\exists$ and $\forall$ is given a game-theoretical interpretation: $\exists x$ marks a move of Player II and $\forall y$ marks a move of Player I in a perfect information zero sum game. We can give meaning also to an infinitary formula

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \phi(x_0, x_1, x_2, \ldots) \tag{1}$$

in a given model by considering the game

$$\begin{array}{c|cccc} \mathrm{I} & x_0 & & x_2 & \ldots \\ \hline \mathrm{II} & & x_1 & & x_3 \ldots \end{array} \tag{2}$$

in which Player II wins if in the end $\phi(x_0, x_1, x_2, \ldots)$ holds. The truth of (1) in a given model is then defined as the existence of a winning strategy for Player II in (2). A winning strategy either exists or does not exist, so we get two valued logic. Note that we could give also the game (2) *itself* as the meaning of (1) in a model, but this would not lead to classical logic.

Henkin's shocking observation is that we can give meaning also to a formula of the form

$$\ldots \exists x_3 \forall x_2 \exists x_1 \forall x_0 \phi(x_0, x_1, x_2, \ldots) \tag{3}$$

although the corresponding game is a game that has a finish but no beginning. Henkin translates the existence of a winning strategy in (2) to the existence of Skolem functions $f_1(x_0), f_3(x_0, x_2) \ldots$ such that

$$\forall x_0 \forall x_2 \ldots \phi(x_0, f_1(x_0), x_2, f_3(x_0, x_2), \ldots) \tag{4}$$

and then observes that in the same vein the obvious meaning of (3) is the existence of functions

$$f_1(x_2, x_4, \ldots), f_3(x_4, x_6, \ldots) \ldots$$

such that

$$\forall x_0 \forall x_2 \ldots \phi(x_0, f_1(x_2, x_4, \ldots), x_2, f_3(x_4, x_6, \ldots), \ldots) \tag{5}$$

Henkin then goes on to observe that the quantifiers need not even be in a linear order, as in (1) and (3), but can have any partial order, and still the method of Skolem functions gives meaning to the quantifier prefix. Henkin goes on: if the quantifiers can be in *any* partial order, even finite partial orders may give rise to interesting new quantifier prefixes, as in

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \phi(x, y, u, v), \tag{6}$$

Is the finite formula (6) first order definable? In 1958 Henkin mentions this problem to Ehrenfeucht, who had come with Mostowski to visit Berkeley, and Ehrenfeucht came up with the sentence

$$\exists z(\psi(z) \wedge \begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} ((x = u \leftrightarrow y = v) \wedge (\psi(x) \rightarrow \psi(y)) \wedge y \neq z), \tag{7}$$

which says that ψ is satisfied by infinitely many elements. Mostowski (10) had just shown that expressing infinity leads not only (obviously) outside first order logic but to a logic that cannot be effectively completely axiomatized.

Enderton (2) and Walkoe (13) vastly extended Ehrenfeucht's result by showing that the expressive power of finite sentences of the form (6), with possibly more rows and columns of quantifiers, is exactly that of existential second order logic. The result holds also in finite models where existential second order logic has the same expressive power as the complexity class NP. This brings quantifiers such as (6), called nowadays *Henkin quantifiers*, right into the heart of descriptive complexity theory and thereby into computer science logic.

As the final step in his analysis of quantifier prefixes Henkin observes that while (3) does not really make sense game theoretically, (6) certainly does. The game behind (6) is simply the imperfect information game in which Player II picks v without knowing x.

Hintikka and Sandu (6) broke the prefix of (6) into pieces by introducing a slash-notation $\exists v/x$ ("there exists v independently of x") which enabled them to write (6) in a linear way as

$$\forall x \exists y \forall u \exists v/x \phi(x, y, u, v, \vec{z}).$$

The slashed quantifiers, such as $\exists v/x$ above, become simply new logical operations to be added to first order logic. The resulting extension of first order logic is called Independence Friendly Logic, in short IF logic, and that is the topic of the book under review. Semantics of IF logic is defined along the lines of the game theoretic semantics of Henkin, appealing to games of imperfect information when necessary. Alternatively one can use Tarski-style semantics, introduced by Hodges (7; 8), in terms of so called teams.

A *team* is a set of assignments. Semantics in IF logic, and other similar logics can be readily defined by extending the usual Tarski-style semantics from single assignments to sets of assignments. The presence of several assignments makes it possible to give meaning to concepts such as independence (4), dependence (12), inclusion and exclusion (3), etc. This has been known in database theory since the 70s and is part and parcel of standard textbooks of database theory. In the case of IF logic the meaning of the slashed quantifier $\exists x/y$ is: Suppose M is a model and X is a team (i.e., a set, or a "database") of assignments into M. We denote by $s(a/x)$ the assignment obtained from the assignment s by changing the value at x to a. With this notation:

$$M \models_X \exists x/y\phi(x, y, \vec{z}) \tag{8}$$

if and only if there is a function $f : X \to M$ such that $M \models_Y \phi(x, y, \vec{z})$ for the team $Y = \{s(f(s)/x) : s \in X\}$ and, in addition,

If $s, s' \in X$ such that $s(u) = s'(u)$ for variables u other than y, then $f(s) = f(s')$. (9)

Thus the quantifier $\exists x/y$ can be further broken into two parts: the first part is the quantifier part $\exists x$ and the second part says that X obeys the dependence condition (9). It is exactly the dependence condition (9) rather than IF logic itself that has a vast supporting literature in database theory (see (11) for an overview). The book under review makes no reference to this, while the related textbook (12) on the topic takes it as the starting point.

By the results of Enderton and Walkoe (op. cit.), sentences of IF logic constitute an alternative syntax for existential second order logic. Several properties of IF logic follow immediately from this: The Compactness Theorem, the Downward Löwenheim-Skolem, the Craig Interpolation Theorem, and the fact that only sentences of IF logic that are equivalent to first order sentences have a negation in the usual sense of the word (the negation symbol is restricted to be used in front of atomic formulas only). Likewise, there is no implication in IF logic.

So what can be done with IF logic, a logic with existential second order power, but no negation, no implication and no effective axiomatization? In the reviewer's view the bigger picture, using games and teams for an investigation of the concepts of dependence and independence in various fields of science and humanities, is a worthy goal. IF logic is part of this bigger picture. The subtitle of the book under review is "A Game-Theoretic Approach" and, indeed, the book is limited to aspects of IF logic with game-theoretic flavour, such as mixed strategies, equilibrium semantics, and perfect recall.

The book starts with two introductory chapters, one on relevant game theory and another on first order logic, and then proceeds to introduce IF logic, its game theoretic semantics and its team semantics (Chapter 4). Elementary (and some less elementary, almost puzzling) properties of IF logic constitute Chapter 5. This chapter is indeed quite useful for those interested in IF logic per se, with all its quirks and twists. The basic connection to existential second order logic is the topic of Chapter 6. The main new material over and above e.g. (12) is contained in Chapter 7 (Probabilistic IF logic) and Chapter 8 (Compositionality, IF modal logic).

As a whole the book is well-written and a valuable source for anyone interested in developments in game theoretic approaches to logic and in the emergent logical study of dependence and independence pheanomena.

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